Electromagnetism in anisotropic chiral media

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We present here for homogeneous anisotropic chiral media an electromagnetic formalism manifestly covariant under the complex rotation group O(3,ℂ). The constitutive relations, the electromagnetic field equations, and the boundary conditions are discussed. To display the main features of this formalism, we discuss the propagation of plane waves in a medium that is anisotropic and chiral but simple enough to make calculations tractable.

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I. INTRODUCTION

Previous works [1,2] on chiral electromagnetism are generalized here to anisotropic media. We use tensor notation with the metric tensor

$$g_{00}=1,\quad g_{ij}=-\delta_{ij},\quad g^{\alpha\beta}=g_{\alpha\beta}\ , \tag{1}$$

where δ_{ij} is the Kronecker symbol. The greek indices take the values 0,1,2,3 and the latin indices the values 1,2,3. We also use the summation convention on repeated indices.

Let $F_{\alpha\beta}$ and $G_{\alpha\beta}$ be the two antisymmetric electromagnetic field tensors that are written, in tensor form,

$$\mathbf{F}_{\alpha\beta}=\begin{vmatrix}0 & E_x & E_y & E_z\\ & 0 & -B_z & B_y\\ & & 0 & -B_x\\ & & & 0\end{vmatrix},\qquad G_{\alpha\beta}=\begin{vmatrix}0 & D_x & D_y & D_z\\ & 0 & -H_z & H_y\\ & & 0 & -H_x\\ & & & 0\end{vmatrix}, \tag{2}$$

where **E**, **H**, **B**, **D** are, respectively, the electric and magnetic fields, the magnetic induction, and the electric displacement. Let $\mathcal{F}^{\alpha\beta}$ be the dual tensor

$$\mathcal{F}^{\alpha\beta}=\tfrac{1}{2}\bar{\epsilon}^{\alpha\beta\gamma\delta}\mathbf{F}_{\gamma\delta}\ , \tag{3}$$

where $\bar{\epsilon}^{\alpha\beta\gamma\delta}$ is the permutation tensor equal to $+1$ (-1) for any even (odd) permutation and zero if any of two indices are equal. Then, from (2) and (3) we get

$$\mathcal{F}^{\alpha\beta}=\begin{vmatrix}0 & -B_x & -B_y & -B_z\\ & 0 & E_z & -E_y\\ & & 0 & E_x\\ & & & 0\end{vmatrix}. \tag{4}$$

From now on, we use E_1, E_2, E_3, and E_j for E_x, E_y, E_z, and similarly for **B**, **D**, **H**. Then, the covariant form of Maxwell's equations is [3]

$$\vartheta_\alpha G^{\alpha\beta}=0,\quad \vartheta_\alpha \mathcal{F}^{\alpha\beta}=0\ , \tag{5}$$

where ϑ_α is the partial derivative operator with respect to x_α.

Equations (5) are covariant under the full Lorentz group L including time and space inversions. So they are not suitable for chiral media that are not invariant under space inversions. Consequently, since relativity requires covariance only under the connected component L_0 of the Lorentz group [4], one has to look for an electromagnetism covariant under L_0 but not under L. As previously discussed [1,2] there exist two groups isomorphic to L_0: on one hand, the three-dimensional (3D) complex rotation group O(3,ℂ) and, on the other hand, the group SL(2,ℂ) of the 2×2 unimodular matrices. We use here O(3,ℂ) and it has been known for a long time [5] that for an homogeneous isotropic achiral medium, the formalism covariant under O(3,ℂ) uses the complex vector

$$\Lambda_j=i\sqrt{\epsilon}E_j+\sqrt{\mu}H_j,\quad i=\sqrt{-1}\ , \tag{6}$$

where ϵ and μ are, respectively, the permittivity and the permeability of the medium. One must notice that the presence of the imaginary unit i is not a matter of convenience; it appears because under space inversions **E** is a polar vector and **H** an axial vector. Consequently, under a space inversion Λ_j transforms into its complex conjugate Λ_j^*. Previously [1] we generalized (6) to homogeneous isotropic chiral media, and here we discuss its extension to anisotropic media.

II. THREE-DIMENSIONAL COMPLEX FORMALISM

A. Constitutive relations

One first has to define $G_{\alpha\beta}$ in terms of $F_{\alpha\beta}$ by using the constitutive relations of the medium. These relations come from outside the theory and in some sense they are arbitrary, except that they have to satisfy the relativistic covariance. This means that in a general *linear* medium the relation between $G_{\alpha\beta}$ and $F_{\alpha\beta}$ takes the form

$$G^{\alpha\beta}=\tfrac{1}{2}\chi^{\alpha\beta\gamma\delta}F_{\gamma\delta}\ , \tag{7}$$

where $\chi^{\alpha\beta\gamma\delta}$ is a complex rank-four tensor which has been thoroughly discussed by Post [6]. This tensor satisfies the relations

$$\chi^{\alpha\beta\gamma\delta}=-\chi^{\beta\alpha\gamma\delta},\quad \chi^{\alpha\beta\gamma\delta}=-\chi^{\alpha\beta\delta\gamma},\quad \chi^{\alpha\beta\gamma\delta}=\chi^{\gamma\delta\alpha\beta}\ . \tag{8}$$

Here we assume $\chi^{\alpha\beta\gamma\delta}$ real and in matrix form one has [6]

$$\begin{bmatrix} & \chi^{\alpha\beta\gamma\delta} & \chi^{\alpha\beta 0j} & \chi^{\alpha\beta jl} \\ & & -E_j & \bar\epsilon^{jlp}B_p \\ \chi^{0k\gamma\delta} & D_k & -E_{kj} & \gamma_{kp} \\ \chi^{lm\gamma\delta} & \bar\epsilon^{lmn}H_n & \gamma_{jn} & \chi_{np} \end{bmatrix} . \tag{9}$$

In the matrix (9) $\bar\epsilon^{lmn}$ is the 3D permutation tensor. The diagonal matrices represent the permittivity ϵ_{jh} and the inverse of the permeability χ_{jh}. The matrix γ_{jh} and its transposed γ_{kj} represent the Fresnel-Fizeau effect [6]. One must note that γ_{jk} and j_{kj} are 3D pseudotensors that are not invariant under space inversions since they connect a polar vector with an axial vector. To avoid confusion with permittivity ϵ, one notes $\bar\epsilon$ the permutation tensors.

B. Electromagnetic field equations

To obtain from (5) and (7) a formalism covariant under O(3,ℂ) we introduce the complex antisymmetric tensor generalizing (6)

$$M^{\alpha\beta}=\mathcal{F}^{\alpha\beta}+iG^{\alpha\beta}\ , \tag{10}$$

that is, according to (3) and (7)

$$M^{\alpha\beta}=\tfrac{1}{2}(\bar\epsilon^{\alpha\beta\gamma\delta}+i\chi^{\alpha\beta\gamma\delta})F_{\gamma\delta}\ . \tag{11}$$

Then the Maxwell equations (5) become $\vartheta_\alpha M^{\alpha\beta}=0$, that is,

$$\vartheta_0 M^{0j}+\vartheta_k M^{kj}=0, \tag{12a}$$

$$\vartheta_j M^{j0}=0\ . \tag{12b}$$

These last equations are not independent: (12b) is a consequence of (12a)since M^{kj} is antisymmetric. Now let P^j and Q_l be the two complex vectors

$$P^j=M^{0j},\quad M^{kj}=i\bar\epsilon^{kjl}Q_l\ . \tag{13}$$

Then Eqs. (12) become

$$\vartheta_0 P^j-i\bar\epsilon^{jkl}\vartheta_k Q_l=0\ , \tag{14a}$$

$$\vartheta_j P^j=0\ , \tag{14b}$$

which are the electromagnetic field equations in the 3D complex formalism. Using (9), (11), and (13) we get for the components of **P** and **Q** the following expressions:

$$\begin{aligned} P^j=M^{0j}&=\tfrac{1}{2}\bar\epsilon^{0jlm}F_{lm}\\ &\quad+\frac{i}{2}(\chi^{0j0l}F_{0l}+\chi^{0jl0}F_{l0}+\chi^{0jkl}F_{kl})\ ,\\ &=B^j+i(\epsilon^{jl}E_l+\gamma^{jl}B_l)\ , \end{aligned} \tag{15}$$

that is

$$P^j=B^j+iD^j\ . \tag{15'}$$

In the same way

$$\begin{aligned} i\bar\epsilon^{kjl}Q_l&=M^{kj}-\tfrac{1}{2}\bar\epsilon^{kjl0}F_{l0}+\frac{i}{2}(\chi^{kjlm}F_{lm}+\chi^{kjl0}F_{l0}\\ &\qquad+\chi^{kj0l}F_{0l})\ ,\\ &=-\bar\epsilon^{kjl}E_l+i(\bar\epsilon^{kjn}\epsilon^{lmr}\bar\epsilon_{lmp}\chi_{nq}B^p+\bar\epsilon_{kjm}\delta_{lm}E^l)\ ,\\ &=\bar\epsilon^{kjl}[-E_l+i(\chi_{lp}B^p+\gamma_{pl}E^p)]\ , \end{aligned}$$

leading to

$$Q_l=\chi_{lp}B^p+\gamma_{pl}E^p+iE_l \tag{16}$$

or

$$Q_l=H_l+iE_l\ . \tag{16'}$$

Under space reflections the vectors **P** and **Q** transform into their complex conjugate.

In the 3D complex formalism there is no need to make a distinction between covariant and contravariant vectors, so that when indices are needed we only use the lower ones. We also write

$$\begin{aligned} \mathbf{P}&=\mathbf{B}+i\mathbf{D}=\nu\mathbf{B}+i\mathbf{D}=\nu\mathbf{B}+i\epsilon\mathbf{E}\ ,\\ \mathbf{Q}&=\mathbf{H}+i\mathbf{E}=\chi\mathbf{B}+i\nu^T\mathbf{E}\ , \end{aligned} \tag{17}$$

with

$$\nu=I+i\gamma\ . \tag{17'}$$

I is the 3×3 identity matrix and ν^T the Hermitian conjugate of ν. Using the nabla symbol ∇ we write the electromagnetic field equations

$$\nabla\times\mathbf{Q}=-i\vartheta_0\mathbf{P}\ , \tag{18a}$$

$$\nabla\cdot\mathbf{P}=0\ . \tag{18b}$$

Now let $\mathbf{\Pi}=\mathbf{\Lambda}+i\mathbf{\Sigma}$ be a complex Hertz vector. Then the solutions of (18) is

$$\mathbf{P}=i\nabla\times\mathbf{\Pi},\quad \mathbf{Q}=\vartheta_0\mathbf{\Pi}\ . \tag{19}$$

Substituting (17) into (19) gives, for the components $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ of $\mathbf{\Pi}$, the system of equations

$$\mathbf{B}=-\nabla\times\mathbf{\Sigma},\quad \mathbf{E}=\vartheta_0\mathbf{\Sigma}\ , \tag{20}$$

and

$$\begin{aligned} \gamma\mathbf{B}+\epsilon\mathbf{E}&=\nabla\times\mathbf{\Lambda}\ ,\\ \chi\mathbf{B}-\gamma^T\mathbf{E}&=\vartheta_0\mathbf{\Lambda}\ . \end{aligned} \tag{21}$$

Taking into account (20) the relations (21) become

$$\begin{aligned} \epsilon\vartheta_0\mathbf{\Sigma}-\gamma\nabla\times\mathbf{\Sigma}&=\nabla\times\mathbf{\Lambda}\ ,\\ \chi\nabla\times\mathbf{\Sigma}+\gamma^T\vartheta_0\mathbf{\Sigma}&=-\vartheta_0\mathbf{\Lambda}\ . \end{aligned} \tag{22}$$

Then, eliminating $\mathbf{\Lambda}$ gives the equation satisfied by $\mathbf{\Sigma}$,

$$\nabla\times(\chi\nabla\times\mathbf{\Sigma})+\epsilon\vartheta_0^2\mathbf{\Sigma}-\gamma\nabla\times\vartheta_0\mathbf{\Sigma}+\nabla\times(\gamma^T\vartheta_0\mathbf{\Sigma})=0\ , \tag{23}$$

which may be considered as a generalized wave equation. Finally substituting (20) into (17) we get as expressions for **P** and **Q**

$$\mathbf{P}=-\nu\nabla\times\mathbf{\Sigma}+i\epsilon\vartheta_0\mathbf{\Sigma}\ ,$$
$$\mathbf{Q}=-\chi\nabla\times\mathbf{\Sigma}+i\nu^T\vartheta_0\mathbf{\Sigma}\ . \tag{24}$$

These relations supply the solution of the electromagnetic field equations (18) in terms of a solution of the generalized wave equation (23).

C. Boundary conditions

To be complete one has still to discuss the boundary conditions on a surface of discontinuity S inside an homogeneous anisotropic chiral medium. Let **n** be the normal to S. Applying the divergence theorem to (18b) and the Stokes theorem to (18a) gives the integral relations

$$\int_\Gamma n_j da=0\ , \tag{25a}$$

$$\int_C Q_j dP_j=-i\int_{\Gamma'}\vartheta_0 P_j n_j \alpha a\ . \tag{25b}$$

In (25a) Γ is a surface enclosing a finite volume containing S. In (25b) C is a closed contour around S, and Γ' is an open surface spanning the contour, da is the elementary area, and dl_j is a line element on the contour.

Then, making Σ and C tend to zero judiciously [3] supplies the boundary conditions on S

$$n_j(P_j-P_j')=0\ , \tag{26a}$$

$$\bar\epsilon_{jkl}n_k(Q_l-Q_l')=0\ . \tag{26b}$$

The primed and unprimed quantities correspond to the electromagnetic fields on both sides of S. Using (17) one sees at once that the relations (26) supply the usual boundary conditions.

Summing up the relations (19), (23), (24), and (26) provides the tools for solving the problem of electromagnetic waves propagating in a homogeneous anisotropic chiral medium. In the next section we apply these relations to the case of plane waves.

III. PLANE WAVES IN HOMOGENEOUS ANISOTROPIC CHIRAL MEDIA

A. Dispersion relation

For a plane wave with all components of the two vector fields **P** and **Q** proportional to $e^{i(k_jx_j+k_0x_0)}, x_0=ct$, Eqs. (18) become

$$ik_0P+\mathbf{k}\times\mathbf{P}=0,\quad \mathbf{k}\cdot\mathbf{P}=0\ , \tag{27}$$

so that all the results of Sec. II hold valid if one changes ϑ_0 and ∇ respectively, into k_0 and **k**. In particular one has

$$\mathbf{P}=-\nu\mathbf{k}\times\mathbf{\Sigma}+i\epsilon k_0\mathbf{\Sigma},\quad \mathbf{Q}=-\chi\mathbf{k}\times\mathbf{\Sigma}+i\nu^T k_0\mathbf{\Sigma}\ , \tag{28}$$

and $\mathbf{\Lambda}$ is a solution of the algebraic system of equations

$$\mathbf{k}\times(\chi\mathbf{k}\times\mathbf{\Sigma})+\epsilon k_0^2\mathbf{\Sigma}-\gamma(\mathbf{k}\times k_0\mathbf{\Sigma})+\mathbf{k}\times(\gamma^T k_0\mathbf{\Sigma})=0\ . \tag{29}$$

This system supplies two components of $\mathbf{\Lambda}$ in terms of the third one when it has a solution, that is, when the determinant of its coefficients is zero. This last condition is

$$G(k_0,k)\equiv\|\bar\epsilon_{ijk}\bar\epsilon_{lmn}k_jk_m\chi_{kl}+\bar\epsilon_{ijk}k_0k_j\gamma_{kn}^T-\bar\epsilon_{jmn}k_0k_m\gamma_{ij}+k_0^2\epsilon_{in}\|=0\ , \tag{30}$$

where $\|A\|$ denotes det A.

The dispersion relation $G(k_0,k)=0$ generally has many solutions corresponding to the different modes able to propagate in the medium. But let us remark that the relation (30) is homogeneous in k_0 and **k**, that is,

$$G(\rho k_0,\rho\mathbf{k})=0 \tag{31}$$

for arbitrary ρ. Hence differentiating with respect to ρ and setting $\rho=1$ we have

$$k_0G_0+k_jG_j=0\ , \tag{32}$$

where G_0 and G_j are the derivatives of G with respect to k_0 and k_j, respectively. Consequently, the velocity of the plane waves is

$$C_j(k)=-G_j/G_0 \tag{33}$$

and we may define a refractive index by the relation

$$n=G_0/(G_1^2+G_2^2+G_3^2)^{1/2}\ . \tag{34}$$

Substituting the solutions of the dispersion relation (30) into (34) gives the index of refraction corresponding to the different modes propagating in the medium. Of course n depends upon the parameters characterizing the constitutive relations, but in the case of dispersive waves n depends also on k.

From (32) and (34) we get

$$k_j=-nk_0G_j/(G_1^2+G_2^2+G_3^2)^{1/2}\ , \tag{35}$$

leading to $k_jk_j=k_0^2n^2$. This last relation proves that for nondispersive waves, the Fourier transform may be used rightfully.

The general dispersion relation (30) is difficult to solve. That is why in the next section we consider a particular anisotropic chiral medium making calculations feasible.

B. Crystal-like medium

We consider a medium in which the matrices χ,ϵ,γ are diagonal. Explicitly

$$\chi_{ij}=\delta_{ij},\quad \gamma_{ij}=\alpha\delta_{ij},\quad \epsilon_{ij}=r_i\delta_{ij},\quad r_i>0\ . \tag{36}$$

δ_{ij} is the Kronecker symbol, α is a real arbitrary parameter, and r_i is positive. Substituting (36) into the dispersion relation (30) we get

$$G(k_0,\mathbf{k})=\|k_0^2 r_k \delta_{ij}+k_i k_j-k^2\delta_{ij}\|=0\ , \tag{37}$$

which is similar to the dispersion relation obtained for wave propagation in a crystal [7] when the structure of the crystal produces directional effects in the dielectric properties. Explicitly (37) writes

$$\begin{aligned}G(k_0,\mathbf{k})\equiv k_0^6 r_1 r_2 r_3 - k_0^4[&r_2 r_3(k_2^2+k_3^2)\\ &+r_3 r_1(k_3^2+k_1^2)\\ &+r_1 r_2(k_1^2+k_2^2)]\\ &+k_0^2 k^2(r_1 k_1^2+r_2 k_2^2+r_3 k_3^2)=0\ .\end{aligned} \tag{38}$$

We further assume, as in our previous works [1,2], that one has a two-dimensional problem in which the electromagnetic field does not depend upon y. This implies $k_2=0$ and (38) reduces to

$$\begin{aligned}G(k_0,k)\equiv k_0^6 r_1 r_2 r_3 &- k_0^4(r_1 r_2 k_1^2+r_2 r_3 k_3^2+r_3 r_1 k^2)\\ &+k_0^2 k^2(r_1 k_1^2+r_3 k_3^2)=0\ ,\end{aligned} \tag{39}$$

with $k^2=k_1^2+k_3^2$. Leaving aside $k_0^2=0$, the solutions of (39) are

$$\begin{aligned}k_0^2=\frac{1}{2r_1 r_2 r_3}[&r_1 r_2 k_1^2+r_2 r_3 k_3^2+r_3 r_1 k^2\\ &\pm(r_1 r_2 k_1^2+r_2 r_3 k_3^2-r_3 r_1 k^2)]\ ,\end{aligned}$$

that is,

$$k_0^2=(k_1^2+k_3^2)/r_2\ , \tag{40a}$$

$$k_0^2=(r_1 k_1^2+r_3 k_3^2)/r_1 r_3\ , \tag{40b}$$

so that two modes can propagate in this crystal-like medium.

Let us now consider the corresponding indices of refraction. One has

$$\begin{aligned}G_0&=2k_0[2k_0^2 r_1 r_2 r_3-r_2(r_1 k_1^2+r_3 k_3^2)-r_3 r_1 k^2]\ ,\\ G_1&=2k_1 r_1[k^2-k_0^2(r_3+r_2)+2k_1(r_1 k_1^2+r_3 k_3^2)]\ ,\\ G_3&=2k_3 r_3[k^2-k_0^2(r_1+r_2)+2k_3(r_1 k_j^2+r_3 k_3^2)]\ .\end{aligned} \tag{41}$$

To obtain these results (39) has been divided by k_0^2. Without any calculation one sees at once that the refractive index for the mode (40a) is $n=(r_2)^{1/2}$. Let us prove this result with (34). Substituting (40a) into (41) gives

$$\begin{aligned}G_0&=2k_0 r_2(k_0^2 r_1 r_3-r_1 k_1^2-r_3 k_3^2)\ ,\\ G_1&=-2k_1 k_0^2 r_1 r_3+2k_1(r_1 k_1^2+r_3 k_3^2)\ ,\\ G_3&=-2k_3 k_0^2 r_1 r_3+2k_3(r_1 k_1^2+r_3 k_3^2)\ ,\end{aligned} \tag{42}$$

so that

$$G_1^2+G_3^2=4k_0^2 r_2(k_0^2 r_1 r_3-r_1 k_1^2-r_3 k_3^2)\ . \tag{42$'$}$$

Substituting (42) and (42′) into (34) gives $n=r_2^{1/2}$. For the mode (40b) the expressions (41) become

$$\begin{aligned}G_0&=2k_0 r_1 r_3(k_0^2 r_2-k^2)\ ,\\ G_1&=2k_1 r_1(k^2-k_0^2 r_2)\ ,\\ G_3&=2k_3 r_3(k^2-k_0^2 r_2)\ ,\end{aligned} \tag{43}$$

and

$$G_1^2+G_3^2=4(k^2-k_0^2 r_2)^2(k_1^2 r_1^2+k_3^2 r_3^2)\ . \tag{43$'$}$$

Substituting (43) and (43′) into (34) gives

$$n=k_0 r_1 r_3(k_1^2 r_1+k_3^2 r_3)^{-1/2}\ , \tag{44}$$

which is characteristic of a dispersive wave. Using a terminology borrowed from crystal optics [7] we may call ordinary and extraordinary waves the electromagnetic fields corresponding to the modes (40a) and (40b), respectively.

C. Plane waves in a crystal-like medium

Using the solutions of the dispersion relation one can now look at the form of the electromagnetic plane waves. With the matrices (36), the Eq. (29) for the imaginary part of the Hertz potential becomes

$$(k_0^2 r_j-k^2)\delta_{ij}\Sigma_j+k_i k_j\Sigma_j=0\ , \tag{45}$$

and still assuming a two-dimensional problem with $k_2=0$ we get from (45)

$$\begin{aligned}&(k_0^2 r_1-k_3^2)\Sigma_1+k_1 k_3\Sigma_3=0\ ,\\ &(k_0^2 r_2-k_1^2-k_3^2)\Sigma_2=0\ ,\\ &k_3 k_1\Sigma_1+(k_0^2 r_3-k_1^2)\Sigma_3=0\ ,\end{aligned} \tag{46}$$

while the expressions (28) for the electromagnetic field vectors **P** and **Q** are

$$\begin{aligned}P_j&=-\beta\bar{\epsilon}_{jlm}k_l\Sigma_m+ik_0 r_j\delta_{jm}\Sigma_m\ \cdot\\ Q_d&=i\beta k_0\delta_{jm}\Sigma_m-\bar{\epsilon}_{jlm}k_l\Sigma_m\ ,\end{aligned} \tag{47}$$

with

$$\beta=1+i\alpha\ . \tag{47$'$}$$

For the ordinary wave the dispersion relation (40a) implies that the solution of (46) if $\Sigma_1=\Sigma_3=0$ and Σ_2 arbitrary so that according to (47) the components of **P** and **Q** are

$$\begin{aligned}&P_i=-i\beta k_3\Sigma_2,\quad P_2=k_0 r_1\Sigma_2,\quad P_3=i\beta k_1\Sigma_2\ ,\\ &Q_1=-k_3\Sigma_2,\quad Q_2=i\beta k_0\Sigma_2,\quad Q_3=k_1\Sigma_2\ .\end{aligned} \tag{48}$$

For the extraordinary wave the solution of (46) is, according to (40b)

$$\Sigma_2=0,\quad \Sigma_3=\frac{-1}{k_1 k_3}(k_0^2 r_1-k_3^2)\Sigma_1\ . \tag{49}$$

Substituting (49) into (47) gives

$$P_1=k_0r_1\Sigma_1,\quad P_2=i\beta\frac{k_0^2}{k_3}r_1\Sigma_1\ ,$$
$$P_3=-\frac{k_0r_3}{k_1k_3}(k_0^2r_1-k_3^2)\Sigma_1\ ,$$
$$Q_1=i\beta k_0\Sigma_1,\quad Q_2=\frac{k_0^2}{k_3}r_1\Sigma_1\ , \tag{50}$$
$$Q_3=-i\beta\frac{k_0}{k_1k_3}(k_0^2r_1-k_3^2)\Sigma_1\ .$$

The relations (48) and (50) where Σ_2 and Σ_1 are some arbitrary constants give the form of the ordinary and extraordinary electromagnetic plane waves in a crystal-like medium. Then one may generate wave packets if one assumes that Σ_2 and Σ_1 are functions of k_0 and $\mathbf{k}$,

$$\mathcal{P}_j(\mathbf{x},x_0)=\int e^{i(k_0x_0+k_jx_j)}P_j(\mathbf{k},k_0)\times\delta G(\mathbf{k},k_0)d\mathbf{k}\,dk_0\ ,$$
$$\mathcal{Q}_j(\mathbf{x},x_0)=\int e^{i(k_0x_0+h_jx_j)}Q_j(k,k_0)\times\delta G(\mathbf{k},k_0)d\mathbf{k}\,dk_0\ . \tag{51}$$

In these integral relations one uses for P_j and Q_j either (48) or (50). The Dirac distribution δG in (51) means that the integration is not carried out in all the four-dimensional space but only inside the volume bounded by the surface $G(k_0,k)=0$.

D. Reflection and refraction of plane waves in a crystal-like medium

Let us now assume that the plane $z=0$ is a surface of discontinuity S between two crystal-like media. We first discuss the reflection and the refraction on S of an ordinary plane wave incident from $z<0$. According to (40a) we define k_1 and k_3 by the relations

$$k_1=k_0r_2^{1/2}\sin\theta,\quad k_3=k_0r_2^{1/2}\cos\theta\ , \tag{52}$$

and leaving aside the factor $e^{ik_0x_0}$ we take as expression of Σ_2 in (48)

$$\Sigma_2=\begin{cases}Ae^{ik_0\sqrt{r_2}(x\sin\theta_i+z\cos\theta_i)} & \text{(incident wave)}\\ Re^{ik_0\sqrt{r_2}(x\sin\theta_r+z\cos\theta_r)} & \text{(reflected wave)}\\ Te^{ik_0\sqrt{r_2'}(x\sin\theta_t+z\cos\theta_t)} & \text{(refracted wave)}\ .\end{cases} \tag{53}$$

A,R,T are, respectively, the amplitudes of the incident, reflected, and refracted waves; $\theta_i,\theta_r,\theta_t$ the angles of incidence, reflection, and refraction, and $r_2^{1/2}$ and $r_2'^{1/2}$ are the indices of refraction on each side of the discontinuity surface.

The kinematic conditions for reflection and refraction at $z=0$ give the usual Descartes-Snell laws

$$\sqrt{r_2}\sin\theta_i=\sqrt{r_2}\sin\theta_r=\sqrt{r_2'}\sin\theta_t\ . \tag{54}$$

The dynamic conditions are supplied by the boundary conditions (26) that are written [since for the plane $z=0$ one has $n_j=(0,0,1)$]

$$P_3=P_3',\quad Q_2=Q_2',\quad Q_1=Q_1'\ . \tag{55}$$

Assuming that the conditions (54) is fulfilled, and using (48), (52), and (53) we get from (55)

$$\beta(r_2)^{1/2}\sin\theta_i(A+R)=\beta'(r_2')^{1/2}\sin(\theta_t)T\ , \tag{56a}$$
$$\beta(A+R)=\beta'T\ , \tag{56b}$$
$$(r_2)^{1/2}[\cos(\theta_i)A+\cos(\theta_r)R]=(r_2')^{1/2}\cos(\theta_2)T\ . \tag{57}$$

Taking into account (54) the relations (56a) and (56b) supply the same condition. Then the relations (56) and (57) supply R and T in terms of the incident amplitude A.

For the extraordinary wave one may define k_1 and k_3 according to (40b) by the relations

$$k_1=k_0r_3^{1/2}\sin\theta,\quad k_3=k_0r_1^{1/2}\cos\theta\ , \tag{58}$$

and similarly according to (53) we take as the expression of in (50)

$$\Sigma_1=\begin{cases}Ae^{ik_0[\sqrt{r_3}\sin(\theta_i)x+\sqrt{r_1}\cos(\theta_i)z]} & \text{(incident wave)}\\ Re^{ik_0[\sqrt{r_3}\sin(\theta_r)x+\sqrt{r_1}\cos(\theta_r)3]} & \text{(reflected wave)}\\ Te^{ik_0[\sqrt{r_3'}\sin(\theta_t)x+\sqrt{r_1'}\cos(\theta_t)z]} & \text{(refracted wave)}\ .\end{cases} \tag{59}$$

The kinematic conditions for reflection and refraction are the same as (54) with r_3 instead of r_2, while using (50), (58), and (59) and assuming the kinematic conditions fulfilled we get for the boundary conditions (55)

$$(r_1r_3)^{1/2}(A\tan\theta_i+R\tan\theta_r)=(r_1'r_3')^{1/2}\tan(\theta_t)T\ , \tag{60a}$$
$$(r_1)^{1/2}\left[\frac{A}{\cos\theta_i}+\frac{R}{\cos\theta_r}\right]=(r_1')^{1/2}\frac{T}{\cos\theta_t}\ , \tag{60b}$$
$$\beta(A+R)=\beta'T\ . \tag{60c}$$

Taking into account the Descartes-Snell laws one sees at once that the relations (60a) and (60b) supply the same condition. Then, from (60b) and (60c) we get R and T in terms of the incident amplitude A.

Remark. A nonchiral crystal medium is obtained for $\alpha=0$, that , is $\beta=1$.

IV. CONCLUSIONS

From a formal point of view the 3D complex electromagnetism is rather simple, reducing in fact in the absence of charges and currents to the generalized wave equation (23). But solving this last equation is rather challenging and will probably require approximate methods. The particular case of plane waves in a crystal-like medium is not likely realistic and its main virtue was to make calculations tractable. Nevertheless, as simple as it is, this example shows the great diversity of waves able to propagate in a medium that is anisotropic and chiral. It also proves that a surface of discontinuity may behave as an opaque screen for some kinds of waves. These results suggest that it is worth thinking about methods to solve consistently the electromagnetic equations of the 3D complex formalism.

[1] A spinor analysis of electromagnetism in anisotropic chiral media has been performed as was made for isotropic media. The spinor formalism is a bit more intricate than the 3D complex formalism since in addition to the spinor fields corresponding to the vector fields P and Q, one needs the spinors corresponding to the complex conjugate vectors $\bar{P},\bar{Q}$ (the spinors are not complex conjugate). But the spinor formalism leads to first-order partial differential equations easily solved in terms of one scalar field when the constitutive relations are real. Applying the spinor formalism to the simple problem discussed here provides of course the same results and in this case the dispersion relations are obtained almost trivially. So the question of the better formalism remains open. The author may provide a copy of his works on the spinor formalism to any interested reader.

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